

# TOPOLOGICAL INSULATORS-TRANSPORT IN CURVED SPACE

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## Abstract

We introduce new methods for investigating propagation of electrons in a multi band system with spin orbit interaction which are time reversal invariant . Therefore Kramer's theorem imposes constraints which give rise to non-trivial Berry connections. As a result chiral zero modes appear at the interface between two domains characterized by parameters which are above or below some critical values. The variation of the parameters is due to disorder, geometry or topological disorder such as dislocations and disclinations. The mechanism might explain the high conductivity coming from the bulk of the topological insulators. We introduced the method of curved geometry and study the effect of dislocations on the Topological Insulators. As a demonstration of the emergent Majorana Fermions we consider a P-wave wire coupled to two metallic rings.

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# I-INTRODUCTION: THE BERRY CONNECTION AND CURVATURE FOR PARTICLES IN SOLIDS

One of the important concepts in Condensed Matter Physics is the idea of Topological order [1–10, 12, 32]. This concept has been confirmed experimentally by the 3D class of Topological Insulators (TI)  $Bi_2Se_3$ ,  $Bi_2Te_3$  and  $Bi_{1-x}Sb_x$  (characterized by a single Dirac cone which lies in a gap)[1, 15, 18, 19]. At the boundary of the 3D Topological insulator one obtains a 2D surface which an odd number of chiral excitations. The two-dimensional  $CdTe/HgTe/CdTe$  quantum wells behave as a 2D TI when the width of the well exceeds 6.3 nm, the  $s$  type  $\Gamma_6$  band to be below the  $p$  type  $\Gamma_8$  [30]. The boundary of the 2D surface is described by a 1D gap less chiral mode. (It is the single chirality which prohibits back-scattering and gives rise to the quantized spin-Hall effect.) Recently the QSH (Quantum Spin Hall effect) [30] in  $CdTe/HgTe/CdTe$  has been discovered [18],[31] and at the edges a spin Hall state dubbed *helical liquid* has been proposed. This spin model is time reversal invariant and therefore obeys Kramer's theorem  $T^2 = -1$  ( $T$  is the time reversal symmetry  $T = -i\sigma^2 K$ ,  $K$  is the conjugation operation and  $\sigma^2$  is a Pauli spin operator). The helical liquid is given by a one dimensional massless chiral fermion: *right* fermions with spin *up* and *left* fermions with spin *down*. The spin Hall Hamiltonian is a physical realization of the lattice model proposed by [2].

In a variety of Physical problems these properties are described by the topological Chern number,  $Ch_n = \int_{T^{D=2n}} Tr[\frac{1}{n!}(\frac{iF}{2\pi})^n]$  where  $T^{D=2n}$  represents the  $D$  dimensional Torus ( the First Brillouin Zone (FBZ) )in  $D$  dimensions and  $F$  represents the curvature given in terms of the ground state eigen-spinors .

In a physical language this means that ground state is protected by a gap which is stable to weak perturbations. At the boundary of such a system one finds an odd number of chiral massless fermions. This is revealed by the presence of a single Dirac cone. In the absence of external magnetic fields a well known theorem **the no go theorem** prohibits the existence of a single Dirac cone. The resolution of this paradox is due to Wilson [5] which showed that by adding terms of the form  $\sum_{i=1}^D [\cos(K_i a) - 1]$  to the Dirac Hamiltonian one can choose the parameter of the model such that one has an odd number of chiral fermions on the boundary. This situation has been realized in the class of materials [15, 19]  $Bi_2Se_3$ ,  $Bi_2Te_3$  and  $Bi_{1-x}Sb_x$ . Theoretically a number of methods have been proposed for

identifying the Chern numbers [3–5] and the *Chern – Simons* coefficients which describe the electromagnetic response of the system. The relation between the space dimension  $D$  and the index  $n$  is  $D = 2n$ . The first Chern number  $n = 1$  exists for  $D = 2$  and is non zero for system with broken time reversal symmetry (Quantum Hall effect) [25, 27]. The second Chern number  $Ch_{n=2}$  is non zero for  $D = 4$  and is not restricted to the case where time reversal symmetry is broken. The time reversal invariants system (*TRI*) in the presence of the spin orbit interaction can give rise to topological insulators *TI*. Recently [7] has proposed a method for computing the topological invariants for non interacting electrons in periodic crystals which are time reversal invariant (*TRI*) and obey the inversion symmetry. For system with inversion symmetry the *TRI* are given in terms of the parity of each pair of the Kramers degenerate occupied bands (at the four ( $D=2$ ) and eight ( $D=3$ ) time -reversal and parity invariant points in the Brillouin zone). A general theory based on *particle-hole* symmetry; *time-reversal* symmetry and their product allows to identify 10 symmetry class [14]. However we believe that the calculation of the invariants for systems which do not have parity invariance can be done, directly by computing the zero modes on the boundary. Using the index theorem we can relate the number of zero modes with the Chern numbers.

Recent experiments point to interesting Quantum oscillations in topological insulators [33] evidence of Landau Quantization in TI [34] and Aharonov-Bohm oscillations for cylinders [35]. Some of the experiments demand the use of geometry in curved space [36, 37, 61]. It seems that topological effects such as dislocations and disclinations might play an important role [16, 39].

Zero modes can appear also at the interface between Superconductors and normal metals. For example the Chiral superconductor  $(p_x + ip_y)$  in the weak pairing limit is given by a similar hamiltonian as considered for the spin Hall effect. The physical difference is that for the superconductor the spinor is built from *particles* and *holes* and **the negative energy of particles describes the same excitations as positive ones**. As a result, the creation and annihilation operators are given by the **Majorana** fermions [40] built from a linear superposition of a fermion with the **charge conjugated** fermion. The edge states for this case will be given by the one dimensional Ising model which has zero modes with a degenerate  $Z_2$  ground state [41] (Majorana zero mode is trapped in each vortex core which leads to a ground state degeneracy of  $2^{n-1}$  in the presence of  $2n$  vortices).

## II-THE METHOD FOR TOPOLOGICAL INSULATORS

In this section we will introduce a set of methods which we will use in this presentation. We will demonstrate our methodology using a simplified version of the spin orbit model studied by [1, 2, 5–8, 10, 12, 13, 15, 17]. We consider a four dimensional lattice in which we introduce a *defect* in one of the direction. We show that for certain parameters a zero mode appears at the interface between domains which have different parameters. When we take into consideration the time reversal points in the Brillouin zone we can introduce an alternative way for determining the criteria of topological insulator (TI). In section B2 we characterize the TI in terms of the Chern numbers. We show that the different Chern numbers are determined by the symmetry group which connect different different region in the Brillouin zone. In section B3 we show that the Hubbard Stratonovici field which results from the electron-electron interaction can be treated as fourth dimensions giving rise to a TI in three dimensions.

Using the momentum space representation in periodic solids [23, 45–47] we have constructed a gauge theory in the momentum space which gives rise to *non-commuting* [9, 20–22] coordinates .

For periodic potentials the eigenvalues are characterized by the Bloch eigenfunctions  $e^{i\vec{K}\cdot\vec{r}}u_{n,\vec{K}}(\vec{q})$  and eigenvalues  $E_n(\vec{K})$  where  $n$  is the band index. In the momentum representation the coordinate  $r^i$  is represented by  $i\partial_{K_i}$ . When more than one atom per unit cell are involved we can have band crossing with linear dispersion (in the vicinity of the crossing points). Due to the band crossing, we can not use a single *matrix connection*  $\langle \vec{K}|d|\vec{K} \rangle \equiv \langle \vec{K}|\partial_{K_i}|\vec{K} \rangle dK^i$  [27] (  $|\vec{K} \rangle$  are the eigenvectors) for the entire first Brillouin zone (FBZ) (which is equivalent to the  $D$  dimensional torus  $T^D$ ). Due to the multiple connection [9, 20, 22, 24, 27, 28] one has to introduce *transition* functions between the different regions in the FBZ. When the number of bands crossing is odd we have an odd number of Dirac cones. For time reversal invariant (TRI) systems (with the the time reversal (TR) operator  $T$ ) we can apply Kramer's theorem  $T^2 = -1$  at the special time reversal points  $\vec{K} = 0$ ,  $\vec{K}^* = G - \vec{K}^*$  (where  $G$  is the reciprocal lattice vector). We find a number of materials which obeys Kramer's theorem with an odd number of Dirac points  $Bi_2X_3$ , ( $X = Se, Te$ ) ,  $Bi_{1-x}Sb_x$  coined Topological Insulators TI. These materials are characterized by strong spin-orbit scattering and have orthorombic crystal structure with the

space group  $D_{3d}^5$  and have a layer structure with five atomic layers per unit cell. Using a projected atomic *Wannier* function we can write a projected Hamiltonian in the basis of the six  $p$  orbitals  $|p_x, \uparrow\rangle$ ,  $|p_y, \uparrow\rangle$ ,  $|p_z, \uparrow\rangle$  and  $|p_x, \downarrow\rangle$ ,  $|p_y, \downarrow\rangle$ ,  $|p_z, \downarrow\rangle$  [17]. Using these orbitals we obtain a six band Bloch Hamiltonian which are further projected to only four bands.

## IIA-The effective model and the emergent chiral modes at the interface between two domains

In order to demonstrate the physics involved we will consider the toy model of two orbital per unit cell :

$$H = i\lambda_{so} \sum_j [(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j) - (b_j^\dagger b_{j+1} + b_{j+1}^\dagger b_j)] + M \sum_j (a_j^\dagger b_j + b_j^\dagger a_j) - t \sum_j [(a_j^\dagger b_{j+1} + a_{j+1}^\dagger b_j)] \quad (1)$$

The eigenvalues are given by:  $\epsilon(\vec{K}) = \pm \sqrt{(2\lambda_{so} \sin K)^2 + (M - 2t \cos K)^2}$ .

This model has the interesting property that for  $M - 2t = 0$  it develops a *Dirac* cone at  $K = 0$  ( at  $M + 2t = 0$  the Dirac cone is at  $K = \pm\pi$ ). For  $M - 2t < 0$  have at  $K = 0$  a gap given by  $2|M - 2t|$ .

Next we extend the one dimensional model to model which has a boundary at  $x = 0$ . We assume that the Hamiltonian parameters obey for  $x < 0$ ,  $(M - 2t)|_{x<0} < 0$  and for  $x > 0$  the parameter are  $(M - 2t)|_{x>0} > 0$ . As a result of the fact that the Hamiltonian for  $x < 0$  is different from the Hamiltonian for  $x > 0$  gives rise to a mid gap state with zero energy which is to a zero mode.

In order to demonstrate that numbers of zero mode is odd and are chiralic we will we will include a spin orbit interaction . We introduce the basis  $|\uparrow\rangle \otimes |1\rangle$ ,  $|\downarrow\rangle \otimes |1\rangle$ ,  $|\uparrow\rangle \otimes |2\rangle$ ,  $|\downarrow\rangle \otimes |2\rangle$ . We will use the Pauli matrix  $\tau$  to describe the band and  $\sigma$  to describe the *spin*. In this basis we can describe the spin orbit interaction by the term:

$$i\lambda_{so} \sum_{i=1}^D \Psi^\dagger(\vec{r})(\sigma^i \otimes \tau_1)\Psi(\vec{r} + \vec{a}_i) + h.c. \quad (2)$$

The energy difference between the two orbitals is represented by

$$M\Psi^\dagger(\vec{r})(I \otimes \tau_1)\Psi(\vec{r}) \quad (3)$$

The hopping energy between the different orbitals is given by

$$-t \sum_{i=1,2,\dots,D} \Psi^\dagger(\vec{r})(I \otimes \tau_1)\Psi(\vec{r} + a_i) + h.c. \quad (4)$$

This Hamiltonian captures the same physics as used by [7, 12, 13]. The Hamiltonian can be rewritten using the *Dirac matrices* with  $\gamma^0$  given in the chiral representation. We have:  $\gamma^1 = \gamma^0(\sigma^1 \otimes \tau_3)$ ,  $\gamma^2 = \gamma^0(\sigma^2 \otimes \tau_3)$ ,  $\gamma^3 = \gamma^0(\sigma^3 \otimes \tau_3)$  and for the fourth direction we use  $\gamma^5$ . We introduce the notations  $\Psi^\dagger(\vec{r})\gamma^0 \equiv \bar{\Psi}(\vec{r})$ ,  $H = \int d^D r \hat{H}(\vec{r}) = \sum_{\vec{K}} \hat{H}(\vec{K})$  :

$$\begin{aligned} \hat{H} = & \sum_{j=1}^{D-1} i\lambda_{SO}(\bar{\Psi}(\vec{r})\gamma^j\Psi(\vec{r} + a_j) - \bar{\Psi}(\vec{r} + a_j)\gamma^j\Psi(\vec{r})) + \lambda_{SO}(\bar{\Psi}(\vec{r})\gamma^5\Psi(\vec{r} + a_4) - \bar{\Psi}(\vec{r} + a_4)\gamma^5\Psi(\vec{r})) + \\ & M\bar{\Psi}(\vec{r})\Psi(\vec{r}) - t \sum_{j=1}^D (\bar{\Psi}(\vec{r} + a_j)\Psi(\vec{r}) + \bar{\Psi}(\vec{r})\Psi(\vec{r} + a_j)) \\ \hat{H}(\vec{K}) = & \bar{\Psi}(\vec{K})[\lambda_{SO} \sum_{j=1}^D \gamma^j \sin(K_j a) + (M - 2t \sum_{j=1}^D \cos(K_j a))]\Psi(\vec{K}) \end{aligned} \quad (5)$$

The eigenvalue of this Hamiltonian are given by:

$\epsilon(\vec{K}) \equiv \pm \sqrt{\sum_{j=1}^D (2\lambda_{SO} \sin(K_j a))^2 + [M - 2t \sum_{j=1}^D \cos(K_j a)]^2}$  where  $D$  is the spatial dimension. For  $\vec{K} \approx 0$  the system has a gap given by  $M - 2tD$  which vanishes at the *critical* value  $\frac{M}{2tD} = 1$  and for  $\vec{K} \approx [\pi, \pi, \pi, \pi]$  the gap occurs at  $M + 2tD$ . In order to investigate the effect of the interface we will assume that in the  $D$  direction (which we take discrete to be  $j$ ) we have a defect region with the parameters  $M(j)$ ,  $t(j)$  and  $\lambda_{SO}(j)$ . For the rest of  $D - 1$  dimensions we have a perfect lattice which obeys periodic boundary conditions. We note that the eigenfunction for such a problem takes the form  $\Omega_{\vec{K}}(\vec{r}, j) = e^{i\vec{K} \cdot \vec{r}} U_s(j)$  where  $U_s(j)$  is a four component spinor,  $\vec{K}$  and  $\vec{r}$  are the momentum and coordinates in  $D - 1 = 3$  dimensions. The spinor  $U_s(j)$  obeys the *zero* eigenvalue equation:

$$\begin{aligned} & \left[ \sum_{i=1}^3 \gamma^i \sin(K_i a) + M(j) - 2t(j) \sum_{i=1}^3 \cos(K_i a) \right] U_s(j) + [t(j)\gamma^5 + \lambda_{SO}(j)] U_s(j+1) - \\ & [t(j-1)\gamma^5 - \lambda_{SO}(j-1)] U_s(j-1) = 0 \end{aligned} \quad (6)$$

We consider a situation where  $M(j) = 2t(j) \sum_{i=1}^3 \cos(K_i a) + \epsilon(j)$  where  $\epsilon(j) = \delta_{j,-1}\epsilon_0 - \delta_{j,1}\epsilon_0$ . To simplify the problem we take  $t(j)$  and  $\lambda_{SO}(j)$  to be constant. The spinor  $U_s(j)$

obeys  $\gamma^5 U_{\pm}(j) = \pm U_{\pm}(j)$ . Since the zero mode must be normalized we expect that only one of the spinors  $U_{\pm}(j)$  will satisfy the zero mode equation. Therefore we have a chiral massless fermion at the interface. This condition can be used for any time reversal points in the Brillouin zone. For  $D = 3$  the 8 time reversal points are given  $\Gamma_{l=n_1, n_2, n_3} = \pi(n_1, n_2, n_3)$  where  $n_i = 0, 1, i = 1, 2, 3$  and the gaps occur at  $\vec{K} = \pi[n_1, n_2, n_3]$ . Using the critical condition  $M = 2t \sum_{i=1}^3 \cos(K_i a)$  we find that for a fixed value of  $\frac{M}{2t}$  the chiral modes will occur at  $\vec{K} = \pi[n_1, n_2, n_3]$  only if  $\frac{M}{2t}$  obeys  $\frac{M}{2t} < [\cos(\pi \cdot n_1) + \cos(\pi \cdot n_2) \cos(\pi \cdot n_3)]$ . Following [7] we identify the value  $\delta_{i(n_1, n_2, n_3)} = -1$  with the points where  $\frac{M}{2t} < [\cos(\pi \cdot n_1) + \cos(\pi \cdot n_2) + \cos(\pi \cdot n_3)]$  and  $\delta_{i(n_1, n_2, n_3)} = 1$  with the points where  $\frac{M}{2t} > [\cos(\pi \cdot n_1) + \cos(\pi \cdot n_2) + \cos(\pi \cdot n_3)]$ . This allows us to make contact with the condition for a strong topological insulator  $(-1)^{v_0} = \prod_{i=1}^8 \delta_{i(n_1, n_2, n_3)}$ , when the product is  $-1$  we have a strong TI. Our condition is applicable to any interface between two domains, all we need to consider is the equation  $M(j) - 2t(j)[\cos(\pi \cdot n_1) + \cos(\pi \cdot n_2) + \cos(\pi \cdot n_3)]$ .

The application of these results to  $D = 3$  will allow to treat the  $D - 1 = 2$  boundary as a two dimensional surface which gives rise to a Helical metal [32]. (The particles ( $\epsilon(\vec{K}) > 0$ ) have an opposite chirality in comparison with the antiparticles (the holes). For a TI in  $D = 2$  dimensions the boundary corresponds to one dimensional chiral fermions where spin up fermions propagate in one direction and spin down electron propagate in the opposite direction [31].

## IIB-The formulation of the TI in terms of the Chern numbers

In order to compute the response of a TI to an external Electromagnetic  $EM$  field we need to know the non-zero Chern numbers which are the coefficients of the Chern-Simons action obtained after integrating the Fermions. For the remaining part we will consider the TI situation with the inverted band condition  $M - 2tD < 0$  at  $\vec{K} \approx 0$ . Using the eigen-spinor we will obtain the transformed coordinate representation, since this can not be done with a single transformation (due to singular points on the torus) we find that the coordinates do not commute and therefore the system has a non zero curvature.

The spectrum of the TI has electron like spinors  $u(\vec{K}, s)$   $s=1,2$  (positive energy) with eigenvalue  $\epsilon(\vec{K}) > 0$  and the antiparticles (hole like) spinors  $v(\vec{K}, s)$   $s = 1, 2$  for  $\epsilon(\vec{K}) < 0$ . This allows to introduce the *electron* like operator  $c(\vec{K}, s)$  and the



*anti - particle* (hole) operator  $b(\vec{K}, s)$  which annihilate the fully occupied ground state  $|G_0\rangle$ ,  $c(\vec{K}, s)|G_0\rangle = 0, b(\vec{K}, s)|G_0\rangle = 0$ . The four component spinor operator  $\Psi(\vec{r}) = [\Psi(\vec{r})_{1,\uparrow}, \Psi(\vec{r})_{1,\downarrow}, \Psi(\vec{r})_{1,\downarrow}, \Psi(\vec{r})_{1,\uparrow}]^T$  has the momentum expansion:

$$\Psi(\vec{r}) = \int_{T^D} \sum_{s=1,2} [c(\vec{K}, s)u(\vec{K}, s)e^{i\vec{K}\cdot\vec{r}} + b^\dagger(-\vec{K}, s)v(-\vec{K}, s)e^{i\vec{K}\cdot\vec{r}}] \quad (7)$$

The ground state eigenspinor is given by  $|v(\vec{K}, s)\rangle$ . Following [9, 21] we find that the transformation  $|\vec{K}, \sigma = \uparrow, \downarrow\rangle \rightarrow |v(\vec{K}, s)\rangle$  replaces the coordinate representation  $r^i = i\partial_{K_i} \rightarrow R^i = i\partial_{K_i} + A^-(\vec{K})_i$  where  $A^-(\vec{K})$  is the connection matrix [9]:

$$\begin{aligned} A^-(\vec{K}) &= \sum_{s=1,2} \sum_{s'=1,2} \langle v(\vec{K}, s) | dv(\vec{K}, s') \rangle |v(\vec{K}, s)\rangle \langle v(\vec{K}, s')| = \\ &\sum_{i=1}^D \sum_{s=1,2} \sum_{s'=1,2} \langle v(\vec{K}, s) | \partial_{K^i} v(\vec{K}, s') \rangle |v(\vec{K}, s)\rangle \langle v(\vec{K}, s')| dK^i = \\ &\sum_{i=1}^D [A_i^{-(0)}(\vec{K})I + \sum_{\alpha=1,2,3} \frac{1}{2} \sigma^\alpha A_i^{-,(\alpha)}(\vec{K})] dK^i \end{aligned} \quad (8)$$

The connection matrix  $A^-(\vec{K})_i$  forms an  $SU(2)$  group. The transformed coordinates obey new commutation rules given by the curvature  $F_{i,j}$ :

$$[R^i(\vec{K}), R^j(\vec{K})] = (-i)F_{i,j} \quad (9)$$

where  $F_{i,j} = \sum_{\alpha=1,2,3} \sigma_\alpha F_{i,j}^\alpha$  and

$$F_{i,j}^\alpha = \partial_{K^i} A_j^{-,(\alpha)}(\vec{K}) - \partial_{K^j} A_i^{-,(\alpha)}(\vec{K}) + \epsilon_{\alpha,\beta,\epsilon} A_i^{-,(\alpha)}(\vec{K}) A_j^{-,(\beta)}(\vec{K}) \quad (10)$$

Following [26] we define the  $n$ 'th Chern number using the notation:  $Trace[\mathbf{F}] \equiv Tr[\mathbf{F}]$  for the curvature  $F_{i,j}(\vec{K})$ .

$$Ch_{n=\frac{D}{2}} = \int_{T^{D=2n}} Tr \left[ \frac{1}{n!} \left( \frac{i\mathbf{F}}{2\pi} \right)^n \right] \quad (11)$$

Using the Heisenberg equation of motion for the  $D$  dimensional TI we find that the Dirac matrices  $\gamma$  are proportional to the velocity operator  $\frac{dR^i(\vec{K})}{dt} \equiv \gamma^i$  [52]. In the presence of weak external fields  $a_\mu(\vec{r}, t)$  we find with the help of the non-commuting coordinates and

the relation  $\frac{dR^i(\vec{K})}{dt} \equiv \gamma^i$  that the response to the external fields (in agreement with [12] is given by:

$$\delta S_{eff.} = Ch_{n=\frac{D}{2}} \Gamma_{C.S.}^D. \quad (12)$$

where

$$\Gamma_{C.S.}^D = \int d^D r \int dt \epsilon^{\mu_1, \mu_2 \dots \mu_{D+1}} a_{\mu_1} \partial_{\mu_2} a_{\mu_3} \dots a_{\mu_{D+1}} \quad (13)$$

### a) The second $Ch_2$ number for TI

We propose to use the theory of first class constraints [29, 43, 44] to derive the second Chern number for the TI. For  $D = 4$  the only non zero Chern number is  $Ch_2$  [26]. The index theorem [26] relates the number of chiral fermions to the Chern number :

$$N_+ - N_- = Ch_{n=\frac{D}{2}} \quad (14)$$

which allows us to make the connection between the number of zero modes of opposite chirality to the  $Ch_{n=\frac{D}{2}}$ . When  $M - 2tD < 0$  at  $\vec{K} = 0$  one finds on the boundary only a single zero mode. According to the index theorem this will imply that the Chern number must be one. The time reversal points give rise to two connections matrices  $A^-(\vec{K})$  and  $A^-(-\vec{K})$ , which are defined on the upper and lower side of the torus  $T^D$  respectively. Each half torus can be mapped to a sphere  $S^D$ . Using homotopy we have the map  $S^4 \rightarrow SU(2)$ . According to [26] we obtain  $\Pi_4[SU(2)] = Z_2$ . We conclude that the Chern number  $Ch_{n=2} = Z_2$ . To compute the Chern number we will use  $A^-(\vec{K})$  for the North sphere (upper torus) and  $A^-(-\vec{K})$  for the South sphere (lower part of the torus). Following [26] we have the identity:  $Tr[(F)^2] = d(Tr[AdA + \frac{2}{3}A^3])$  ( $(F)^2$  is *closed* but not *exact* (we can not satisfy the identity with one connection  $A$  on the entire torus ). Therefore the  $Ch_2$  number will be given by a difference of *two* three form Chern-Simons terms:

$$Tr[(F)^2] = dTr[AdA + \frac{2}{3}A^3] \quad (15)$$

Using the connections  $A^-(\vec{K})$  and  $A^-(-\vec{K})$  we find:

$$\begin{aligned}
Ch_2 &= \int_{T^{D=4}} Tr[\frac{1}{2!}(\frac{i\mathbf{F}}{2\pi})^2] = \int_{S^4-North} Tr[\frac{1}{2!}(\frac{i\mathbf{F}_{North}}{2\pi})^2] + \int_{S^4-South} Tr[\frac{1}{2!}(\frac{i\mathbf{F}_{South}}{2\pi})^2] \\
&= \frac{1}{8\pi^2} \int_{S^3} (Tr[(A^{-,North}(\vec{K}))^3] - Tr[(A^{-,South}(\vec{K}))^3]) = \\
&\frac{1}{8\pi^2} \int_{S^3} (Tr[(A^{-}(\vec{K}))^3] - Tr[(A^{-}(-\vec{K}))^3]) = \frac{1}{24\pi^2} \int_{S^3} [(g^{-1}dg)^3] = Z_2
\end{aligned} \tag{16}$$

The difference between the two connections  $A^{-}(\vec{K}) - A^{-}(-\vec{K}) = g^{-1}dg$  defines the transition function. At the TRI points [7] the time reversal operator  $T = -i\sigma_2 \otimes IK$  defines the matrix elements  $\langle -\vec{K}, s|T|\vec{K}, s' \rangle = g_{s,s'}(\vec{K}) = -g_{s',s}(-\vec{K})$ . This relation imposes a constraint on the connection field. The solution of this constraint follows from the *BRST* Cohomology [29]. The solution of the constraint will be that  $g^{-1}dg$  is an element of the  $Z_2$  group therefore  $\frac{1}{24\pi^2} \int_{S^3} [(g^{-1}dg)^3]$  takes two values 0 or 1.

### b)The first Chern numbers for D=2

Following the theory of the Chern numbers for  $D = 2$  the first Chern number is non-zero if time reversal symmetry is violated. Therefore the proposal [30] that the Spin Hall conductance is quantized for the two dimensional heterostructure *CdTe/HgTe/CdTe* needs clarifications.

#### b1)- $Ch_1$ in the absence of time reversal symmetry for D=2

We consider the Hamiltonian:

$$h(\vec{K}) = K^1\sigma_2 - K^2\sigma_2 + M(\vec{K})\sigma_3 \tag{17}$$

When  $M(-\vec{K}) = M(\vec{K})$  the time reversal symmetry is broken ( the TR operator is given by  $T = -i(I \otimes \sigma^2)\mathbf{K}$ , where  $\mathbf{K}$  is the conjugation operator).

For this case we have a single eigenstate for the ground state  $|v(\vec{K})\rangle$ . Using this eigenstate we compute the *connection one form* [26]  $A^{-}(\vec{K}) = \langle v(\vec{K})|dv(\vec{K})\rangle$ . Due to the singularity at  $\vec{K} = 0$  [26] we map the upper torus  $T^2$  to the *North* sphere  $S^2$  defining the north connection  $A^{-(North)}(\vec{K})$  and the lower part of the torus  $T^2$  is mapped to the south sphere with the South connection  $A^{-(South)}(\vec{K})$ . The North eigenvector is related to the South eigenvector through a gauge transformation. When we rotate the eigenstate  $|v(\vec{K})\rangle$  by  $2\pi$  the state is modified by  $e^{i\pi}$  ( note the factor  $\frac{1}{2}$  which is due to the spin half system). Therefore we have the relation between the north and south state :

$$|v(\vec{K})\rangle_{North} = e^{i\frac{\varphi}{2}} |v(\vec{K})\rangle_{South} \quad (18)$$

which gives:

$$\begin{aligned} Ch_1 &= \int_{T^{D=2}} Tr[\frac{1}{1!}(\frac{iF}{2\pi})] = \int_{S_2} Tr[\frac{1}{1!}(\frac{iF}{2\pi})] = \\ &= \frac{1}{2\pi} \oint_S i[A^{-(North)} - A^{-(South)}] = \frac{1}{2\pi} \oint_S \frac{1}{2} d\varphi = \frac{1}{2} \end{aligned} \quad (19)$$

This implies that the Hall conductance for the Dirac equation is  $\frac{e^2}{2h}$  instead of  $\frac{e^2}{h}$ . In the presence of a boundary which restricts the system to  $x > 0$  the term  $M(\vec{K})\sigma_3$  will give rise to a *single spin Polarized* massless fermion.

### **b2)-The first Chern number for time reversal system when D=2**

Following [30] and the experimental discovery [18, 19] the Hamiltonian for the two dimensional heterostructure  $CdTe/HgTe/CdTe$  with the thickness  $d_{Q.W.}$  exciding  $6.3 \text{ nm}$  has inverted bands and is given by :

$$h(\vec{K}) = K^1\tau^2 \otimes I - K^2\tau^1 \otimes \sigma^3 + [M - 2t(\cos(K_1a) - 2t\cos(K_2a))]\tau^3 \otimes I \quad (20)$$

For  $\vec{K} \approx 0$  the condition for TI is  $\frac{M}{4t} < 1$ . This model has the special feature that for spin *up* and spin *down* we have two Hamiltonians:

$$h^\uparrow(\vec{K}) = K^1\tau^2 - K^2\tau^1 + [M - 2t(\cos(K_1a) - 2t\cos(K_2a))]\tau^3 \quad (21)$$

$$h^\downarrow(\vec{K}) = K^1\tau^2 + K^2\tau^1 + [M - 2t(\cos(K_1a) - 2t\cos(K_2a))]\tau^3 \quad (22)$$

Therefore both Hamiltonians can be diagonalized separately in the band space  $\tau$ . In each band the time reversal operator (does not include the spin operator) is given by  $T_\tau = \mathbf{K}$  and obeys  $T_\tau^2 = 1$ . Since  $h^\uparrow$  and  $h^\downarrow$  (separately) are *not time reversal invariant*. We conclude that in the  $\tau$  space each Hamiltonian has  $Ch_1$  number denoted by  $Ch_1^\uparrow$  and  $Ch_1^\downarrow$ . Following the discussion presented above we have:

$$[R^i(\vec{K}), R^j(\vec{K})] = f_{ij}^3 \tau^3 \otimes \sigma^3 \quad (23)$$

Therefore the Chern number will be an integer for each spin polarization:

$$Ch_1^\uparrow = \int_{TD=2} Tr[\frac{1}{1!}(\frac{iF}{2\pi})] = \int_{S_2} Tr[\frac{1}{1!}(\frac{iF}{2\pi})] = \frac{1}{2\pi} \oint_S i[A^{-(North)} - A^{-(South)}] = \frac{1}{2\pi} \oint_S d\phi = 1 \quad (24)$$

Contrary to the previous case (b1) where the transition function between the North and the South pole was  $\frac{1}{2}d\phi$ , for the present case the transition function will be  $d\phi$  (not half)! Repeating the calculation for  $Ch_1^\downarrow$  we find  $Ch_1^\downarrow = -1$ . Therefore the spin Hall conductance will be given by  $2 \cdot e^2 h$  characterized by the first spin Chern number  $Ch_1^{spin} = \frac{1}{2}[Ch_1^\uparrow - Ch_1^\downarrow]$ .

**IIC-Time Reversal Invariant Topological Insulators in 3 + 1 dimensions induced by electron-electron interactions reproduce effectively the theory in 4 + 1 dimensions**

The external fields are restricted to three dimensions. Using the effective action  $\delta S_{eff.} = Ch_2 \Gamma_{C.S.}^{D=4}$  we will take the fourth dimension field  $a_{\mu=4}(\vec{r}, t)$  arbitrarily. We compactify the fourth dimension  $\phi(\vec{x} = x^1, x^2, x^3, t) = \oint dx^4 a_{\mu=4}(\vec{r} = x^1, x^2, x^3, x^4, t)$  [12] and obtain :

$$\delta S_{eff.}^{(3)} = Ch_{n=2} \cdot \phi(\vec{x}, t) \int d^3x \int dt \epsilon^{\mu, \nu, \sigma, \tau} \partial_\mu a_\nu(\vec{x}, t) \partial_\sigma a_\tau(\vec{x}, t) \quad (25)$$

There are only two values  $\theta \equiv Ch_{n=2} \cdot \phi(\vec{x}, t)$  which respect the time reversal symmetry,  $\theta = \pi$  corresponds to a TI and  $\theta = 0$  correspond to a regular insulators.

**We introduce a construction which is entirely in three dimensions space and is generated by the electron-electron interactions**

The Hubbard-Stratonovich transformation allows the representation :

$$\sum_a \rho_a(\vec{r}) \bar{\Psi}(\vec{r}) \gamma^a \Psi(\vec{r}) + (\rho_a(\vec{r})) \frac{1}{2\kappa(\vec{r}, \vec{r}')} (\rho_a(\vec{r}')) \quad (26)$$

When the channel  $\rho_5(\vec{r}) \gamma^5$  dominates the two body interaction we can use single variable  $q \equiv \rho_5(\vec{r})$ . We obtain a four dimensional model build from the three dimensional momentum  $K_1, K_2, K_3$  with the new field (due to interactions)  $q$ . This allows us to use the  $D = 4$

formulation with the vector  $\vec{K} = [K_1, K_2, K_3, q]$ . (For an arbitrary interaction we can construct a four dimensional vector in the following way:

$$\vec{K} = [K_1 + \rho_1, K_2 + \rho_2, K_3 + \rho_3, \rho_5] \quad (27)$$

Following the procedure from the previous section we compute the second Chern number:

$$Ch_2 = \frac{1}{32\pi^2} \int_{T^4} d^3K \int_{q_{min.}}^{q_{max.}} dq \epsilon^{i,j,k,l} Trace[F_{i,j} F_{k,l}] \quad (28)$$

Due to the integration measure  $(\rho_a(\vec{r})) \frac{1}{2\kappa(\vec{r}, \vec{r}')} (\rho_a(\vec{r}'))$  we approximated the range  $q = \rho_5(\vec{r})$  to  $q_{min.} < q < q_{max.}$ .

In order to compute the polarization we must consider the external electromagnetic interactions  $h^{EM}$ :

$$h^{EM} = e\Phi(\vec{r}, t) + \frac{e}{2} \left[ \frac{d\vec{r}}{dt} \cdot \delta \vec{A}^{(M)}(\vec{r}, t) + \delta \vec{A}^{(M)}(\vec{r}, t) \cdot \frac{d\vec{r}}{dt} \right] \quad (29)$$

For a weak electric and magnetic field in the the  $z$  direction we introduce  $E_3^{ext.}(\vec{Q}) = -\frac{\dot{A}_3^{ext.}(\vec{Q})}{iQ_3}$  and  $A_2^{ext.}(\vec{Q}) = \frac{r_1 B_3^{ext.}(\vec{Q}_2, \dots)}{iQ_2}$

We substitute the transformed coordinates  $r^i \rightarrow R^i$  The derivative of the coordinate is related to the commutator  $\frac{dR^i}{dt} = \dots E_3^{ext.}[R^i, R^3], i = 1, 2$ . We substitute the expression for  $\frac{dR^i}{dt}$  into the magnetic interaction  $\frac{e}{2} \left[ \frac{d\vec{r}}{dt} \cdot \delta \vec{A}^{(M)}(\vec{r}, t) + \delta \vec{A}^{(M)}(\vec{r}, t) \cdot \frac{d\vec{r}}{dt} \right]$  using a slowly varying magnetic field,  $A_2^{ext.}(\vec{Q}) = \frac{r_1 B_3^{ext.}(\vec{Q}_2, \dots)}{iQ_2}$ . Keeping terms to first order in  $Q$  we find that the interaction can be written as the nonlinear polarizability similar to the *axion electrodynamics* [11, 12, 60].

$$\frac{1}{32\pi^2} \int_{q_{min.}}^{q_{max.}} dq \int_{T^3} d^3K \epsilon^{i,j,k,l} Tr(< [R^i, R^j][R^k, R^l] >) E_3(\vec{Q}) B_3(\vec{Q}) \equiv \vartheta \frac{e^2}{2\pi h} \int d^3x \int dt \vec{E} \cdot \vec{B} \quad (30)$$

where the expectation value  $< [R^i, R^j][R^k, R^l] >$  is taken with respect to the interaction measure  $\sum_a (\rho_a(\vec{r})) \frac{1}{2\kappa(\vec{r}, \vec{r}')} (\rho_a(\vec{r}'))$ .

### III- TRANSPORT ON THE BOUNDARY OF A TOPOLOGICAL INSULATOR

The three dimensional Topological Insulators  $Bi_2Se_3, Bi_2Te_3, Sb_2Te_3$  support on their two dimensional boundary (surface) chiral helical metals described by a two component spinor. When the space vector  $\vec{r}$  is rotated by an angle  $\phi$  the transformation  $\vec{r} \rightarrow \vec{r}'$  generates a spinor transformation  $\Psi(\vec{r}) \rightarrow \Psi'(\vec{r}') = e^{i\phi \cdot \frac{\sigma_3}{2}} \Psi(\vec{r})$ . This implies that when we perform a rotation of  $\phi = 2\pi$  the spinor accumulates a Berry phase of  $\pi$  [9]. This effect suggests that in the presence of *disorder* the state  $\Psi(\vec{r})$  is scattered to a new state  $\Psi'(\vec{r}')$ . When the scattered state is returned back to the point  $\vec{r}$  it will have an extra Berry phase of  $\pi$ . Since localization effects are determined by the interference between the closed path and a time reversal path, the extra phase of  $\pi$  will give rise to destructive interference, which means *absence* of localization. This idea has not been confirmed experimentally. The main difficulty being the fact that the experiments use nontrivial geometries such as cylinders or spheres. The *topology* affects the Dirac equation by adding new *unknown* Berry phases. The Aharonov -Bohm oscillations experiment performed in nano-wires  $Bi_2Se_3$  in [35] seems to support the idea that the maximum conductance should occur for a zero or integer flux  $\Phi = n \frac{hc}{e}$  (in zero flux we have a Berry phase of  $\pi$ ) and the minimum should be at  $\Phi = (n + \frac{1}{2}) \frac{hc}{e}$  where localization is supposed to take place. The experiment of [35] has been performed in a cylindrical geometry. On a cylinder the coordinate transformation (from the plan to the cylinder) modifies the boundary conditions for the spinors from **periodic to anti-periodic boundary conditions** [36]. As a result the Berry phase is modified by an extra half unit flux. Therefore, when an external flux pierces the cylinder we obtain gapless excitation which propagate along the  $z$  only when the flux is exactly  $\frac{1}{2} \frac{hc}{e}$ . This means that the maximum of the conductance will occur for fluxes  $\Phi = (n + \frac{1}{2}) \frac{hc}{e}$  and the minimum should be for zero flux or  $\Phi = (n) \frac{hc}{e}$ .

Applying an external magnetic field to the two dimensional surface reveals that the Landau quantization and the Hall conductance are modified on the boundary of TI. Due to the Dirac form the Hall conductance is given by  $\sigma_{x,y}^{surface} = \frac{e^2}{h} (n + \frac{1}{2})$ . Recent experiments of scanning tunneling spectroscopy (STS) [35] reveal the existence of a state at  $n = 0$  which is not affected by the value of the magnetic field (the Landau levels for  $n \geq 1$  scale with the magnetic field  $\sqrt{B}$ ). The experiments are performed on curved geometries [35] and therefore the quantization rules are modified. For example the Hall conductance for the Dirac equation on a **sphere** is  $\sigma^{sphere}(x, y) = 2 \cdot \sigma^{surface}(x, y) = (2n + 1) \frac{e^2}{h}$  [36]. This suggests that for a thin sample  $\sigma^{thin}(x, y) = \sigma^{top}(x, y) + \sigma^{bottom}(x, y)$  (the measurement might show contribution

from the *top* and from the *bottom* surface). This suggests that for a thin sample the Hall conductance will be similar to the Hall conductance on a sphere.

The momentum-resolved Landau spectroscopy of the Dirac surface state in  $Bi_2Se_3$  [35] reveals in addition to the Landau spectroscopy, triangular-shaped defects which are caused by the presence of vacancies of  $Bi$  defects at the  $Se$  sites. Clearly the scanning tunneling microscopy indicates a modification of the local densities.

These experiments suggest that the Dirac equation is sensitive to defects and geometry and therefore modified quantization rules are expected and probably revealed through the new Berry indices  $\gamma \neq \frac{1}{2}$ . (In a recent Shubnikov-de Haas oscillations experiment performed on the topological Insulator  $Bi_2Se_3$  the authors present data which show magnetic oscillations which correspond to the Landau level quantization  $E_n = v_F \sqrt{2(n + \gamma)\hbar e B}$  where  $\gamma$  is the Berry phase which is different from  $\gamma = \frac{1}{2}$ .) The appearance of the new indices  $\gamma \neq \frac{1}{2}$  might be due to topology of the defects or/and interactions.

The transport results for the conductance of the TI  $Bi_{1-x}Sb_x$  show interesting oscillations in the presence of a magnetic field as a function of the angle between the axes  $C_3$  and  $C_2$  [33]. The transport experiments  $Bi_{1-x}Sb_x$  performed in the range  $0.7 < x < 0.22$  show a significant contribution to the conductivity which comes from the *bulk*.

It seems that the results are not ruling out the surface conduction contribution, but suggests contribution from the *bulk*. The theory of TI predicts that when a magnetic field perpendicular to the two dimensional surface is applied the conductance  $\sigma_{x,x}$  *decreases* since the Berry phase which in the absence of magnetic field is  $\pi$ , becomes in a magnetic field  $\pi(1 \pm \frac{\Delta}{\sqrt{(\Delta)^2 + |K|^2}})$ . Therefore backscattering is allowed ( The parameter  $\Delta$  contains two contributions, the normal component of the external magnetic field and the magnetic exchange which is due to the *magnetic impurities*. Both effects can be described by a modification of the Hamiltonian by  $\delta H = \Delta \cdot \sigma^3$ . It is known that in a system with spin orbit interaction the conductivity also decreases in the presence of a magnetic field [42]. Therefore a quantitative analysis is needed in order to separate the two effects. If the Topological Insulators are to be responsible for the conductivity one needs to understand the surface-bulk contributions.

### IIIA- The Effect of Static Disorder on the Second Chern Number



The transport experiments [33] performed on the mixed crystal  $Bi_{1-x}Sb_x$  TI for  $0.7 < x < 0.22$  show that a significant contribution to the conductivity comes from the bulk. In the absence of disorder the condition  $\frac{M}{2tD} < 1$  guarantees the existence of the TI. The presence of disorder, is included through the impurity potential:  $V_{disorder} = V_0(\vec{r})\gamma^0 + V_1(\vec{r})I$  ( $V_0(\vec{r})$  represents the space dependent mass term - the orbital splitting energy). The mass gap  $M$  is normalized by both terms.

### a)-The *two mode* crystal

For inhomogeneous disorder we have a *two mode* crystal, namely the crystal splits into *domains* where the mass gap  $M$  takes values,  $\frac{M_{eff}(A)}{2tD} < 1$  (domain A) and  $\frac{M_{eff}(B)}{2tD} > 1$  (domain C). At the interface between the domains we will have *chiral* massless fermion propagation. The recent transport experiment [33] hints that a significant contribution to the conductivity comes from the bulk. Therefore the *two mode* proposal might explain the high conductivity coming from the domain walls in the bulk.

### b)-The homogeneous *virtual* crystal.

When the system is homogeneously disordered, the value of the mass gap can be replaced by  $\sqrt{\langle M^2 \rangle} = \hat{M}$  plus Gaussian fluctuations of  $M - \hat{M}$ . The value  $\frac{\hat{M}}{2tD} = 1$  represents the transition point between the TI and a regular insulator. Due to the Gaussian fluctuations we expect the following transitions: The region  $\frac{\hat{M}}{2tD} < 1 - \epsilon$  is a TI. The region  $1 - \epsilon < \frac{\hat{M}}{2tD} < 1 + \epsilon$  corresponds to a chiral metal which will become a regular insulator for  $\frac{\hat{M}}{2tD} > 1 + \epsilon$ .

In order to perform a quantitative analysis we will consider the Hamiltonian  $H_{disorder} = H + V_{disorder}$ . Using the eigen-spinors  $|v(\vec{K}, s)\rangle, |u(\vec{K}, s)\rangle$  we compute the transformed disorder matrix  $\hat{V}_{s,s'}(\vec{K}, \vec{K}')$  (with the matrix elements  $V_{s,s'}^{c^\dagger, c}(\vec{K}, \vec{K}')$ ,  $V_{s,s'}^{b^\dagger, b}(-\vec{K}, -\vec{K}')$ ,  $V_{s,s'}^{c^\dagger, b^\dagger}(\vec{K}, -\vec{K}')$ ,  $V_{s,s'}^{b^\dagger, c^\dagger}(-\vec{K}, \vec{K}')$  computed with the help of the Fourier transform of the disorder potential and the eigen-spinors).

$$H = \int_{T^{D=4}} \sum_{s=1,2} [\epsilon(\vec{K})(c^\dagger(\vec{K}, s)c(\vec{K}, s) + b^\dagger(\vec{K}, s)b(\vec{K}, s))] \quad (31)$$

$$V_{disorder} = \int_{T^{D=4}} \sum_{s=1,2} \int_{T^{D=4}} \sum_{s'=1,2} (c^\dagger(\vec{K}, s), b(\vec{K}, s)) [\hat{V}_{s,s'}(\vec{K}, \vec{K}')] (c(\vec{K}', s'), b^\dagger(\vec{K}', s'))^T \quad (32)$$

We will compute the *self energy* and we will extract the effective mass gap  $M$ . Perturbation theory can be used since time reversal symmetry guarantees that the matrix elements  $\hat{V}_{s,s'}(\vec{K}, \vec{K}' = -\vec{K})$  are zero. In the presence of disorder the ground state spinor  $|v(\vec{K}, s) >$  is replaced by  $|\mathbf{v}(\vec{K}, s) >$ :

$$\begin{aligned}
|\mathbf{v}(\vec{K}, s) > &= |v(\vec{K}, s) > + \sum_{s'=1,2} \sum_{\vec{P}} \frac{|V(\vec{K}, \vec{P})v^*(\vec{K}, s)v(\vec{P}, s')|^2}{\epsilon(\vec{K}) - \epsilon(\vec{P})} |v(\vec{P}, s') > + \\
&\sum_{s'=1,2} \sum_{\vec{P}} \frac{|V(\vec{K}, \vec{P})v^*(\vec{K}, s)u(\vec{P}, s')|^2}{\epsilon(\vec{K}) - \epsilon(\vec{P})} |u(\vec{P}, s') > \equiv \\
|v(\vec{K}, s) > &+ \sum_{s'=1,2} \sum_{\vec{P}} B^{(s,s')}(\vec{K}, \vec{P}) |v(\vec{P}, s') > \\
&+ \sum_{s'=1,2} \sum_{\vec{P}} C^{(s,s')}(\vec{K}, \vec{P}) |u(\vec{P}, s') >
\end{aligned} \tag{33}$$

In order to test the possible phase transition for the Chern number, we propose to compute the new curvature  $\mathbf{F}$  which replaces  $\mathbf{F}$ . For this purpose we introduce the coordinate  $X^i = i\partial_{K^i}$  and compute the *new* connection:  $\mathbf{A}_i^-$ .

$$\begin{aligned}
\mathbf{A}_i^- &= \langle \mathbf{v}(\vec{K}, s_1) | X^i | \mathbf{v}(\vec{K}, s_2) \rangle = \langle v(\vec{K}, s_1) | X^i | v(\vec{K}, s_2) \rangle + \\
&\sum_{s'=1,2} \sum_{s''=1,2} \sum_{\vec{P}} (B^{(s_1,s')}(\vec{K}, \vec{P}))^* X^i B^{(s_2,s'')}(\vec{K}, \vec{P}) \\
&+ \sum_{s'=1,2} \sum_{s''=1,2} \sum_{\vec{P}} (C^{(s_1,s')}(\vec{K}, \vec{P}))^* X^i C^{(s_2,s'')}(\vec{K}, \vec{P}) \dots
\end{aligned} \tag{34}$$

Using this representation we obtain the curvature matrix:  $\mathbf{F}_{i,j} = [X^j, \mathbf{A}_i^-]$ .

Using the new curvature  $\mathbf{F}_{i,j}$  we obtain the second Chern number:

$$Ch_2 = \int_{T^{D=4}} Tr < [\frac{1}{2!} (\frac{i\mathbf{F}}{2\pi})^2] >_{average-disorder} \tag{35}$$

In order to answer the question if the elastic disorder destroys the TI we will perform analytical and numerical calculation. In order to modify the Chern number by disorder the representation for the new eigenvector  $|\mathbf{v}(\vec{K}, s) >$  must be divergent! When the series converges we find that  $[X^j, \mathbf{A}_i^-] = [X^j, A^-(\vec{K})_i]$  and the second Chern number is unchanged.

In order to determine the transport properties we will construct an effective non-linear sigma model using the Keldish formalism [48].

### III.B. The TI with Disorder Using the Keldish Method

In order to compute the conductivity we need to construct an effective non-linear sigma model using the Keldish formalism. Following [48] we introduce two fermionic fields which reside on the forward and backward contour. We replace the *four* component spinor  $\Psi(\vec{r}, \omega)$  by a *eight* component spinor  $\Psi(\vec{r}, \omega) = [\Psi(\vec{r}, \omega), \Psi^\dagger(\vec{r}, -\omega)]^T$ . Using this formalism we represent the disorder action  $S_{disorder}$ :

$$S_{disorder} = \int d^D r \int d\omega V_{disorder}(\vec{r}) (\Psi^\dagger(\vec{r}, \omega) \Sigma_3 \Psi(\vec{r}, \omega)) \quad (36)$$

Performing the average over disorder we can introduce a set of Hubbard Stratonovich fields [50, 58]  $Q_j(\vec{r}, (\omega, \omega'))$  which act as gauge fields in the space  $\Sigma_3 \otimes \gamma^j$  ( $\Sigma_3$  acts on the frequency space). Using the Hamiltonian given in section IIA (the TI in  $D$  dimensions) + disorder we have:

$$\begin{aligned} S = & \int d^D r \int d\omega \int d\omega' [ \sum_{j=1, \dots, D} \Psi^\dagger(\vec{r}, \omega) (\Sigma_3 \otimes \gamma^j) ((-i) \hat{\partial}_j \\ & + Q_j(\vec{r}, \omega, \omega') \Psi(\vec{r}, \omega') + \Psi^\dagger(\vec{r}, \omega) (\Sigma_3 \otimes \gamma^0) ((M + Q_0(\vec{r}, \omega, \omega') \Psi(\vec{r}, \omega') \\ & - t \sum_{j=1, \dots, D} \Psi^\dagger(\vec{r}, \omega') (\Sigma_3 \otimes \gamma^j) \hat{\nabla}_j^2 \Psi(\vec{r}, \omega') ] \end{aligned} \quad (37)$$

where  $\hat{\nabla}_j^2 \Psi(\vec{r}, \omega) \equiv [\Psi(\vec{r} + a_j, \omega) + \Psi(\vec{r} - a_j, \omega) - 2\Psi(\vec{r}, \omega)]$  and the Gaussian gauge fields are controlled by:  $H_Q = \int d^D r \int d\omega \int d\omega' [\sum_{j=1,5} \frac{1}{2V^j} (Q_j(\vec{r}, (\omega, \omega'))^2]$ .

This effective action is used to construct the effective action for transport. Integrating the fermions we obtain the effective action in terms of the gauge fields  $Q_\nu$ . The effective action is a function of the curvature  $F_{\nu, \mu}(\omega, \omega') = \partial_\nu Q_\mu(\vec{r}, \omega, \omega') - \partial_\mu Q_\nu(\vec{r}, \omega, \omega')$  and Chern-Simon term.

Using the *Renormalization Group* we will establish the phase diagram: In the metallic phase the Maxwell term vanishes. If the coefficient of the chern number obeys,  $Ch_{n=2} \neq 0$  we have a chiralic metal. When  $Ch_{n=2} = 0$  we have a regular metal. When the Maxwell term is non zero we have either a regular insulator  $Ch_{n=2} = 0$  or a T.I for  $Ch_{n=2} \neq 0$ .

### III.C. Interaction and Disorder

For the one dimensional boundary TR symmetry forces the backscattering term to vanish. Since *umklapp* is allowed [31] the presence of the Coulomb interactions for  $D = 1$  drives the system to the localized phase. We have found that the Luttinger parameters decreases below  $\frac{1}{4}$  [51, 52] causing the chiral edge excitations to localize. The two dimensional surface represents the boundary of the three dimensional Topological Insulator which is described as  $D = 2$  chiral metal given by [17]:  $h(\vec{K}) = v(K^2\sigma_1 - K^1\sigma_2)$  where  $\vec{K}$  is the momentum and  $\vec{\sigma}$  are the Pauli matrices for the spin of the electron. This Hamiltonian has two eigen-spinors  $|u(\vec{K})\rangle = [u_\uparrow(\vec{K}), u_\downarrow(\vec{K})]^T$  for particles and  $|v(\vec{K})\rangle = [v_\uparrow(\vec{K}), v_\downarrow(\vec{K})]^T$  for antiparticles which have opposite chirality. The two component spinor *operator* is given by:  $\Psi(\vec{r}) = [\Psi(\vec{r})_\uparrow, \Psi(\vec{r})_\downarrow]^T$ . Using the ground state  $|G_0\rangle$  we introduce the *particles* and *anti-particles* operators:  $c(\vec{K})|G_0\rangle = 0$ ,  $b(\vec{K})|G_0\rangle = 0$ . For this case we observe that the spinor operator is given by the chiral Weyl form:

$$\Psi(\vec{r}) = \int_{T^2} [c(\vec{K})u(\vec{K})e^{i\vec{K}\cdot\vec{r}} + b^\dagger(-\vec{K})v(-\vec{K})e^{i\vec{K}\cdot\vec{r}}] .$$

The experiments show that the Dirac Cone is below the conduction band. This can be considered by introducing a positive chemical potential  $\mu > 0$ . The presence of the chemical potential changes the scaling dimensions of the operators and can cause the impurity and interaction potentials to become *relevant* variables (in the sense of the Renormalization Group analysis). At the edge points of the Fermi sphere  $\vec{k}_F = [0, \pm|\vec{k}_F|]$ ,  $\vec{k}_F = [\pm|\vec{k}_F|, 0]$  the electronic spectrum is quadratic! (The excitations in the vicinity of the Fermi surface are given by  $\epsilon(\vec{q}) = \frac{\vec{k}_F}{|\vec{k}_F|} \cdot \vec{q} + \frac{q^2}{2|\vec{k}_F|}$  where  $\vec{q}$  is a measure with respect the Fermi energy.)

### IIID- Topological Insulators in Curved Spaces

It seems that in curved spaces caused by geometry or topological disorder such as dislocations and disclinations we can have the following scenario. For a TI we dislocations might create boundaries such that chiral excitations might be bound to the dislocations. On the other hand if we have already massless fermions some of the topological defects might cause localization.

In order to investigate the effect of the Topological Disorder on the TI we will introduce a geometric formulation for curved spaces [36, 37, 37, 54, 55, 55–57, 61]. We consider an  $n$  dimensional manifold where a mapping from the curved space  $X^a$ ,  $a = 1, 2, \dots, n$  to the *local*

*flat* space exists,  $x^\mu$ ,  $\mu = 1, 2, \dots, n$ . We introduce the tangent vector [37]  $e_\mu^a(\vec{x}) = \frac{\partial X^a(\vec{x})}{\partial x^\mu}$  which satisfies the orthonormality relation  $e_\mu^a(\vec{x})e_\nu^b(\vec{x}) = \delta_{a,b}$  (here we use the convention that we sum over indices which appear twice). Representing the curved coordinate in terms of local flat coordinates gives rise to the metric tensor  $e_\mu^a(\vec{x})e_\nu^a(\vec{x}) = g_{\mu,\nu}(\vec{x})$ . We introduce the linear connection which is determined by the Christoffel tensor  $\Gamma_{\mu,\nu}^\lambda$  which is defined by:

$$\nabla_{\partial_\mu} \partial_\nu = -\Gamma_{\mu,\nu}^\lambda \partial_\lambda \quad (38)$$

Next we introduce the vector field  $\vec{V} = V^a \partial_a = V^\mu \partial_\mu$  and the spinor  $\Psi(\vec{x})$ . The covariant derivative of the vector field  $V^a$  is determined by the spin connection  $\omega_{i,b}^a$ :

$$D_\mu V^a(\vec{x}) = \partial_\mu V^a(\vec{x}) + \omega_{a,b}^\mu V^b \quad (39)$$

Similarly in the basis  $x^\mu$  the covariant derivative  $D_\mu V^\nu(\vec{x})$  is determined by the Christoffel tensor:

$$D_\mu V^\nu(\vec{x}) = \partial_\mu V^\nu(\vec{x}) + \Gamma_{\mu,\nu}^\lambda V^\lambda \quad (40)$$

The spinor connection  $\omega_{a,b}^\mu$  determines the covariant derivative  $D_\mu V^a(\vec{x})$  defined above and as well determine the covariant derivative of the spinor  $\Psi(\vec{x})$  given in term of the gamma matrices  $\gamma$ .

$$D_\mu \Psi(\vec{x}) = (\partial_\mu + \frac{1}{8} \omega_\mu^{a,b} [\gamma^a, \gamma^b]) \Psi(\vec{x}) \quad (41)$$

We first evaluate the Christoffel tensor which is a function of the metric tensor  $g_{\mu,\nu}(\vec{x})$ .

$$\Gamma_{\mu,\nu}^\lambda = -\frac{1}{2} \sum_{\tau=1,2,\dots,n} g^{\lambda,\tau}(\vec{x}) [\partial_\nu g_{\mu,\tau}(\vec{x}) + \partial_\mu g_{\nu,\tau}(\vec{x}) - \partial_\tau g_{\mu,\nu}(\vec{x})] \quad (42)$$

The relation between the spin connection and the linear connection can be obtained from the fact that the two covariant derivative of the vector  $\vec{V}$  are equivalent.

$$D_\mu V^a = e_\nu^a D_\mu V^\nu \quad (43)$$

Since we have the relation  $V^a = e_\nu^a V^\nu$  it follows from the last equation

$$D_\mu [e_\nu^a] = D_\mu \partial_\nu e^a = (D_\mu \partial_\nu) e^a + \partial_\nu (D_\mu e^a) = 0 \quad (44)$$

Using the definition of the Christoffel index we find the relation between the spin connection and the linear connection:

$$D_\mu[e_\nu^a] = \partial_\mu e_\nu^a(\vec{x}) - \Gamma_{\mu,\nu}^\lambda e_\lambda^a(\vec{x}) + \omega_{\mu,b}^a e_\nu^b(\vec{x}) \equiv 0 \quad (45)$$

From this equation we determine the spin connection:

$$\begin{aligned} \omega_\mu^{a,b} &= \frac{1}{2} e^{\nu,a} (\partial_\mu e_\nu^b - \partial_\nu e_\mu^b) - \frac{1}{2} e^{\nu,b} (\partial_\mu e_\nu^a - \partial_\nu e_\mu^a) \\ &\quad - \frac{1}{2} e^{\rho,a} e^{\sigma,b} (\partial_\rho e_{\sigma,c} - \partial_\sigma e_{\rho,c}) e_\mu^c \end{aligned} \quad (46)$$

This allows to rewrite the Dirac equation  $h = \gamma^a(-i\partial_a)$  as :

$$h^{curved} = e_a^\mu \cdot \gamma^a(-i)(\partial_\mu + \frac{1}{8}\omega_\mu^{ab}[\gamma_a, \gamma_b]) \quad (47)$$

For the mapping of a surface to another curved surface the transformation is simplified . For the surface case  $n = 2$  the square of the length element  $[dx^2 + dy^2]$  on a flat surface is related to the square length element  $ds^2$  for a curved surface in the following way:  $ds^2 = e^{-\beta(\vec{x})}[dx^2 + dy^2]$  where  $\beta(\vec{x})$  is the conformal transformation from the flat to curve surface. This method has been used in reference [36] to compute the form of the Dirac equation for a sphere.

### c)The effect of Topological disorder on the Topological Insulators- Dislocations and Disclinations-Computation of the connection.

The presence of *dislocations* and *disclinations* deform substantially the lattice structure and therefore might affect the electronic transport. Following [39] the deformation of the crystal give rise to a coordinate transformation which will affect the Dirac equation. Given a perfect lattice with coordinates  $x^i$ , lattice deformation will modify the coordinate to  $X^a(\vec{x}) = x^a + e^a(\vec{x})$  where  $e^a(\vec{x})$  is the local lattice deformation. As a result the metric is given by  $g_{\mu,\nu}(\vec{x}) = \frac{\partial X^a}{\partial x^\mu} \frac{\partial X^a}{\partial x^\nu}$ .

**Screw dislocations** are characterized by the *Burger* vector  $\vec{B}$  which is parallel to the dislocation line contrary to the edge dislocation where the Burger vector is perpendicular to the dislocation line . The metric tensor for a screw dislocation is :  $ds^2 = (dz + \frac{B^z}{2\pi}d\theta)^2 + d\rho^2 + \rho^2 d\phi^2$ .

Using the tangent vector we represent the burger vector in term of the *torsion*:

$$\oint dX^\mu e_\mu^a = \int \int dX^\mu dX^\nu (\partial_\mu e_\nu^a - \partial_\nu e_\mu^a) = -B^{(a)} \quad (48)$$

In order to describe *disclination* which has no torsion but has *curvature*. The metric tensor for a disclination is given by  $ds^2 = dz^2 + d\rho^2 + \alpha^2 \rho^2 d\phi^2$ . The effect of disclinations can be understood as an angle deficit  $2\pi \cdot \alpha$ , therefore once a rotation of  $2\pi$  is performed the spinor will rotate by an angle  $\pi \cdot \alpha < \pi$ . Therefore we obtain a Berry phase of  $\pi \cdot \alpha$  which will allow backscattering and will allow localization. (Quantitatively the effect of the line deficit can be expressed as the strength of a vortex.) We believe that the dislocation might play an important role for computing transport properties, contrary to regular disorder some of the dislocations and disclinations can cause localization.

#### **d) An explicit demonstration of the method presented for an edge dislocation in two dimensions**

We consider the two dimensional chiral Dirac equation in the absence of any dislocation:

$$h = \sigma^1(-i\partial_2) + \sigma^2(i\partial_1) \quad (49)$$

We consider an edge dislocation with the burger vector  $B^{(2)}$  (the value of the vector is the lattice constant in the  $y$  direction).

The dislocation is given in terms of the *torsion* tensor  $T_{\mu=1,\nu=2}^{a=2}$

$$T_{\mu=1,\nu=2}^{a=2} = \partial_1 e_2^2 - \partial_2 e_1^2 = B^2 \delta^2(\vec{r}) \quad (50)$$

Using the solution of the Laplace equation we obtain the tangent components:

$$e_1^2 = \beta \frac{y}{(x^2 + y^2)} \quad e_2^2 = 1 - \beta \frac{x}{(x^2 + y^2)} \quad (51)$$

where  $\beta = \frac{B^{(2)}}{2\pi}$  and  $e_1^1 = 1, e_2^1 = 0$ .

Using the tangent components we obtain the metric tensor *to first order* in  $\beta$ :

$$g_{11} = 1, \quad g_{12} = \beta \frac{y}{x^2 + y^2}, \quad g_{22} = 1 - \beta \frac{y}{x^2 + y^2}, \quad g_{21} = 0 \quad (52)$$

Using the the vectors components  $e_\mu^a$  and the results obtained in previous equations, we compute the transformed Pauli matrices:

$$\sigma^1(x) \approx \sigma^1, \quad \sigma^2(x) \approx \sigma^2 - \beta \frac{y}{x^2 + y^2} \sigma^1 \quad (53)$$

The spin connection is obtained from equation 46 – 47:

$$\omega_1^{12} = -\omega_2^{12} \approx \pi \beta \delta^2(\vec{x}) \quad (54)$$

As a result the chiral Dirac equation in the presence of an edge dislocations becomes:

$$h^{edge-dislocation} = \sigma^1 \left[ -i\partial_2 - \frac{1}{8}\pi\beta\delta^2(\vec{x})\sigma^3 \right] + \left[ \sigma^2 - \beta \frac{y}{x^2 + y^2} \sigma^1 \right] \left[ i\partial_1 + \frac{1}{8}\pi\beta\delta^2(\vec{x})\sigma^3 \right] \quad (55)$$

Using the new Dirac Hamiltonian in the presence of an edge dislocation we obtain the following eigenvalue equation for the two component spinor,  $E$  is the energy,  $\mu$  is the chemical potential and  $\beta \equiv \frac{8}{\pi} \hat{g}_{edge}$ .

$$(E - \mu)U_{\uparrow}(\vec{r}) = [(\partial_x - i\partial_y) - e^{-\frac{i\pi}{4}} \hat{g}_{edge} \delta^2(\vec{r})]U_{\downarrow}(\vec{r}) - i\frac{8}{\pi} \hat{g}_{edge} \left( \frac{y}{x^2 + y^2} \right) \partial_x U_{\downarrow}(\vec{r}),$$

$$(E - \mu)U_{\downarrow}(\vec{r}) = -[(\partial_x + i\partial_y) + e^{\frac{i\pi}{4}} \hat{g}_{edge} \delta^2(\vec{r})]U_{\uparrow}(\vec{r}) - i\frac{8}{\pi} \hat{g}_{edge} \left( \frac{y}{x^2 + y^2} \right) \partial_x U_{\uparrow}(\vec{r}).$$

The presence of the complex coupling constants  $e^{\frac{\pm i\pi}{4}} \hat{g}_{edge}$  restricts the solutions. Preliminary results indicate that one dimensional propagating solution along the line  $y = -x$  exists, and in addition we The interesting one dimensional solutions confined to the dislocation line is interesting since it shows that the conductance will be dominated by one dimensional channels confined to the edge dislocation.

Therefore for a number of dislocations centered at points  $[x_i, y_i]$   $i = 1, \dots, N$  we will have  $N$  one-dimensional conducting channels confined to the line  $(y(x) - y_i) = -(x - x_i)$ .

### e) The Chiral metallic state induced in a Topological insulator by a Dislocation

Using the coordinate transformation induced by the dislocations we can rewrite the model given in section *IIA*. The new model will contain the Dirac equation in 4+1 dimensions *curved space* (due to dislocations) plus the non relativistic mass term,  $M\bar{\Psi}(\vec{r})\Psi(\vec{r}) - t \sum_{j=1}^D (\bar{\Psi}(\vec{r} + a_j)\Psi(\vec{r}) + \bar{\Psi}(\vec{r})\Psi(\vec{r} + a_j))$  which in the continuum will be replaced by the Laplacean  $\nabla^2 = \Delta$  which in the curved space will become the **Laplace-Beltrami** operator  $\Delta_{LB}$  [56, 57]. The dislocation is believed to give rise to a *chiral zero mode* (along the dislocation line). The eigenvalue equation for the topological insulator in the presence of a dislocation can reveal this fact which has to be further investigated.



#### IV. Topological $Z_2$ Realization for the $p_x + ip_y$ Chiral Superconductor Wire Coupled to Metallic Rings in an External Flux

The experimental realization of the  $p_x + ip_y$  pairing order parameter (p-wave superconductors) in  $Sr_2RuO_4$ ,  ${}^3He - A$  and the  $\nu = \frac{5}{2}$  case represents interesting new excitations. In these (weak coupling)  $p_x + ip_y$  pairing the excitations are half vortices which are zero mode energy Majorana fermions. We consider a p-wave superconductor confined to a one-dimensional wire. At the edges of the wire  $x = 0$  and  $x = L$  the pairing order parameter vanishes and two zero modes appear at the edges ( $x=0$  and  $x=L$ ). Due to the charge conjugation of the Bogoliubov spectrum these zero modes should be Zero mode Majorana Fermions [53]. Mapping the problem to the Ising model one can show that the ground state is a  $Z_2$  doubly degenerate ground state. As a result the single particle excitations are non-local. **As a concrete example we couple the p-wave wire to two rings which are pierced by external fluxes.** In the first stage we will study a spinless model and in the second stage we will consider the effect of spin.

The model for the spinless case is  $H = H_{P-W} + H_1 + H_2 + H_{(P-W,1)} + H_{(P-W,2)}$ .

$H_{P-W}$  is the p-wave superconductor with the pairing gap  $\Delta$  and polarized fermion operator  $C(x) \equiv C_{\sigma=\uparrow}(x)$ .  $H_{P-W} = -t \sum_{x,x'} (C^+(x)C(x') + h.c.) + \lambda \sum_x C^+(x)C(x) - \Delta \sum_{x,x'} [\gamma_{x,x'} C^+(x)C^+(x') + \gamma_{x,x'} C(x')C(x)]$ . The time reversal and parity symmetry are broken,  $\gamma_{x,x'=x-a} = -\gamma_{x,x'=x+a}$ . Following [53] one introduces the Majorana fermions  $\eta_1(x)$ ,  $\eta_2(x)$  [53] and finds:  $C(x) = \eta_1(x) + i\eta_2(x)$ ;  $C^+(x) = \eta_1(x) - i\eta_2(x)$  and  $\eta_1(x) = \frac{1}{2}(C(x) + C^+(x))$  and  $\eta_2(x) = \frac{1}{2i}(C(x) - C^+(x))$ . The  $H_{P-W}$  Hamiltonian has the pairing boundary conditions  $\Delta(x=0) = \Delta(x=L) = 0$  and  $\Delta(x) = \Delta_0$  for  $0 < x < L$ . This model is equivalent to the 1d-Ising model in a transverse field [53]:  $H_{P-W} = \frac{i}{2} \int dx \hat{\eta}^T [\sigma_3 \partial_x + i\Delta(x)\sigma_2] \hat{\eta}$  where  $\hat{\eta}^T = (\eta_1(x), \eta_2(x))$ .

**The two eigenfunctions are charge conjugated with two bound states at  $x=0$  and  $x=L$ .**

$$\eta_1(x) = \eta_L \Phi^L(x) + \dots \text{ and } \eta_2(x) = \eta_0 \Phi^0(x) + \dots$$

$\Phi^0(x)$  and  $\Phi^L(x)$  are the bound states at  $x=0$  and  $x=L$  and  $\eta_0$  and  $\eta_L$  are the zero modes Majorana operators. They obey  $(\eta_0)^+ = \eta_0$ ,  $(\eta_0)^2 = \frac{1}{2}$  and  $(\eta_L)^+ = \eta_L$ ,  $(\eta_L)^2 = \frac{1}{2}$ . This suggests that the ground state is  $Z_2$  doubly degenerate.

Using the zero modes Majorana Fermions  $\eta_0$  and  $\eta_L$  we obtain a fermion operator  $q =$

$\frac{1}{2}(\eta_L + i\eta_0)$  and  $q^\dagger = \frac{1}{2}(\eta_L - i\eta_0)$  which obey anti-commutation relations  $[q, q^\dagger]_+ = 1$  which obeys  $q^\dagger|0\rangle = |1\rangle$  and  $q|1\rangle = |0\rangle$ . The p-Wave wire is described by the ground state  $|0\rangle$  when the wire contains only pairs. When one electron is added to the wire the state is  $|1\rangle$ . The energy of this state is given by  $\epsilon$ . The value of  $\epsilon$  depends on the length of the wire, when  $L \rightarrow \infty$  we have  $\epsilon \rightarrow 0$  [53].

$H_1 + H_2$  represent the Hamiltonian for the two metallic rings each pierced by a flux  $\hat{\varphi}_i$ ,  $i = 1, 2$  [22].

$$H_i = \int_0^{l_{ring}} dx \left[ \frac{\hbar^2}{2m} \psi_i^\dagger(x) (-i\partial_x - \frac{2\pi}{l_{ring}} \hat{\varphi}_i)^2 \psi_i(x) \right] \quad i = 1, 2,$$

In the long wave approximation the  $p$ -Wave Hamiltonian  $H_{P-W}$  is replaced by  $H_{P-W} = \epsilon q^\dagger q$ .

$H_{(P-W,1)} + H_{(P-W,2)}$  represents the coupling between the  $p$ -Wave wire to the two rings through the matrix element  $g$ .

$$H_{(P-W,1)} + H_{(P-W,2)} = ig[q^\dagger(\psi_1^\dagger + \psi_1 + \psi_2^\dagger - \psi_2) + q(\psi_1^\dagger + \psi_1 - \psi_2^\dagger + \psi_2)]$$

Using the Heisenberg equation of motion we substitute  $q(t)$  and  $q^\dagger(t)$  and reduce the problem to a eigenvalue problem in terms of the rings spinless electron operators  $\psi_1(x)$ ,  $\psi_2(x)$ . We will work with the Fourier transform in the frequency  $z$  representation. We replace the two electron operators by the four component spinor,

$$\Psi(x, z) = [\psi_1(x, z), \psi_2(x, z), \psi_2^\dagger(x, -z), \psi_1^\dagger(x, -z)]^T.$$

$$z \begin{pmatrix} \psi_1(x, z) \\ \psi_2(x, z) \\ \psi_2^+(x, -z) \\ \psi_1^+(x, -z) \end{pmatrix} = \quad (56)$$

$$\begin{bmatrix} e_1 & 0 & 0 & 0 \\ 0 & e_2 & 0 & 0 \\ 0 & 0 & -e_2 & 0 \\ 0 & 0 & 0 & -e_1 \end{bmatrix} \begin{pmatrix} \psi_1(x, z) \\ \psi_2(x, z) \\ \psi_2^+(x, -z) \\ \psi_1^+(x, -z) \end{pmatrix} + \frac{2g^2\delta(x)}{\epsilon^2 - z^2} \begin{bmatrix} -z & \epsilon & \epsilon & z \\ \epsilon & -z & -z & -\epsilon \\ \epsilon & -z & -z & -\epsilon \\ z & -\epsilon & -\epsilon & -z \end{bmatrix} \begin{pmatrix} \psi_1(z) \\ \psi_2(z) \\ \psi_2^+(-z) \\ \psi_1^+(-z) \end{pmatrix}.$$

We use the notation:  $e_i = (-i\partial_x + \varphi_i)^2$ .

A full solution involves a numerical computation for the eigenvalues and eigenvectors. We find that the problem simplifies in two limits  $\epsilon \gg g$  and  $\epsilon \ll g$ .

a) When  $\epsilon \ll g$  the current in each ring will depend only on the flux applied. For equal fluxes we will find that the current and the magnetization are proportional to the applied flux. This case corresponds to the *Andreev* reflection at each interface.

b) When  $\epsilon \gg g$  the current through the current and the magnetization will be proportional to the  $(flux)^2$ . This case corresponds to a *crossed Andreev reflection*

We believe that measuring the current and the magnetization as a function of the flux will allow an indirect confirmation for Majorana fermions.

## V-CONCLUSION

This article aims to introduce a variety of powerful geometrical methods for condensed matter systems such as connection and curvature. Electrons in solids are characterized by Bloch bands. When the band crosses one obtains non-trivial phases which gives rise to non-commutativity in momentum space characterized by the Chern numbers. The Topological insulators are material with spin-orbit interaction. At special point in the Brillouin zone the model is time reversal invariant. As a result a new type of insulators characterized by the second Chern number is identified. At the boundary of the Topological Insulators an odd number of Chiral states are identified. Those states are robust against disorder and represent perfect metals.

The idea of protected boundary can be generalized to any interface including disorder or Topological defects such as dislocations and disclinations.

We introduce Majorana fermions and show that they can be detected by measuring the persistent current in metallic rings attached to a P-wave superconductor.

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